Exercise 1. Let $X_1, X_2, \ldots$ be independent and identically distributed continuous random variables. We say that a record occurs at time $n, n > 0$ and has value X_n if $X_n > \max(X_1, \cdots, X_{n-1})$, where $X_0 \equiv -\infty$. (a) Let N_n denote the total number of records that have occurred up to (and including) time n. Compute $E[N_n]$ and $\mathrm{Var}(N_n)$.

(b) Let $T = \min\{n : n > 1 \text{ and a record occurs at } n\}$. Compute $P\{T > n\}$ and show that $P\{T < \infty\} = 1$ and $E[T] = \infty$.

(c) Let T_y denote the time of the first record value greater than y. That is,

$$T_y = \min\{n : X_n > y\}.$$

Show that T_y is independent of X_{T_y}. That is, the time of the first value greater than y is independent of that value. (It may seem more intuitive if you turn this last statement around.)

Solution 1.

(a):

Denote $Y_i = \begin{cases} 1, \text{if record occurs at time } i \\ 0, \text{else} \end{cases}$.

Since $P(Y_i = 1) = P(X_i > \max\{X_1, \ldots, X_{i-1}\})$, then we can have order statistics $X_{(1)}, \ldots, X_{(i-1)}$, which we can consider as a permutation of $X_1, \ldots, X_n$.

Hence we have i interval: $(-\infty, X_{(1)}), [X_{(1)}, X_{(2)}), \ldots, [X_{(i-1)}, \infty)$, and $X_i > \max\{X_1, \ldots, X_{i-1}\}$ if and only if $X_i \in [X_{(i-1)}, \infty)$.

Note that the probability X_i falling into each interval is the same, for $X_{(1)}, \ldots, X_{(i-1)}$ is a permutation of $X_1, \ldots, X_n$.

Thus $P(Y_i = 1) = \frac{1}{i}$, and Y_i are mutually independent.

Then we have

$$\begin{aligned} E[N_n] &= E\left[\sum_{i=1}^{n} Y_i\right] \\ &= \sum_{i=1}^{n} E[Y_i] \\ &= \sum_{i=1}^{n} P(Y_i = 1) \\ &= \sum_{i=1}^{n} \frac{1}{i} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Var}(N_n) &= \sum_{i=1}^{n} \mathrm{Var}(Y_i) \\ &= \sum_{i=1}^{n} \frac{1}{i}\left(1 - \frac{1}{i}\right) \end{aligned}$$

(b):

$P(T > n) = P(\max\{X_1, \ldots, X_n\} = X_1) = P(X_{(n)} = X_1) = \frac{1}{n}$.

$P(T < \infty) = \lim_{n \to \infty} P(T \leq n) = \lim_{n \to \infty}\left(1 - \frac{1}{n}\right) = 1$.

By layer cake representation, we have $E[T] = \sum_{i=0}^{n} P(T > n) = \sum_{i=1}^{n} \frac{1}{i} = \infty$.

(c):

$T_y = \min\{n: X_n > y\} \Longrightarrow X_1, \ldots., X_{T_y - 1} \leq y$

On the other hand, we can also easily have $X_1, \ldots., X_{T_y - 1} \leq y \Longrightarrow T_y = \min\{n: X_n > y\}$.

Thus $T_y \in \sigma(X_1, \ldots., X_{T_y - 1})$ which is independent of X_{T_y}.

Exercise 2. Let X denote the number of white balls selected when k balls are chosen at random from an urn containing n white and m black balls. Compute $E[X]$ and $\mathrm{Var}(X)$.

Solution 2.

Let X_i denote the ith sampling result without replacement, then $X_i = \begin{cases} 1, \text{white ball} \\ 0, \text{black ball} \end{cases}$.

Thus we can decomposite X into k parts, that is, $X = \sum_{i=1}^{k} X_i$.

Note that $P(X_i = 1) = \frac{n}{n+m}$.

Then we can have

$$\begin{aligned} E[X] &= E\left[\sum_{i=1}^{k} X_i\right] \\ &= \sum_{i=1}^{k} P(X_1 = 1) \\ &= \sum_{i=1}^{k} \frac{n}{n+m} \\ &= \frac{kn}{n+m} \end{aligned}$$

and

$$\begin{aligned} \mathrm{Var}(X) &= E[X^2] - [E[X]]^2 \\ &= \sum_{i=1}^{k} E[X_i^2] + 2\sum_{1 \leq i < j \leq k} X_i X_j - \left(\frac{kn}{n+m}\right)^2 \\ &= \sum_{i=1}^{k} E[X_i] + 2\sum_{1 \leq i < j \leq k} P(X_i = X_j = 1) - \left(\frac{kn}{n+m}\right)^2 \\ &= \frac{kn}{n+m} + 2\sum_{1 \leq i < j \leq k} P(X_j = 1 | X_i = 1) P(X_i = 1) - \left(\frac{kn}{n+m}\right)^2 \\ &= \frac{kn}{n+m} + 2\frac{k!}{2!(k-2)!} \cdot \frac{n-1}{n+m-1} \cdot \frac{n}{n+m} - \left(\frac{kn}{n+m}\right)^2 \\ &= \frac{kn(n+m)(n+m-1) + k(k-1)n(n-1)(n+m) - k^2n^2(n+m-1)}{(n+m)^2(n+m-1)} \\ &= \frac{kn(n+m)[(n+m) - 1 + (k-1)(n-1) - kn] + k^2n^2}{(n+m)^2(n+m-1)} \\ &= \frac{knm(n+m-k)}{(n+m)^2(n+m-1)} \end{aligned}$$

Exercise 3. A round-robin tournament of n contestants is one in which each of the $\binom{n}{2}$ pairs of contestants plays each other exactly once, with the outcome of any play being that one of the contestants wins and the other loses. Suppose the players are initially numbered $1, 2, \ldots, n$. The permutation $i_1, \cdots, i_n$ is called a Hamiltonian permutation if i_1 beats i_2, i_2 beats $i_3, \cdots$, and i_{n-1} beats i_n. Show that there is an outcome of the round-robin for which the number of Hamiltonians is at least $n!/2^{n-1}$.

Solution 3.

Denote $X_{(i_1, \ldots, i_n)} = \begin{cases} 1, \text{if } (i_1, \ldots, i_n) \text{ is Hamiltonian} \\ 0, \text{else} \end{cases}$. Since Hamiltonian permutation exists, then

$$\begin{aligned} E\left[\sum_{\substack{(i_1, i_2, \ldots i_n) \\ \text{is a permutation of } (1, \ldots, n)}} X_{(i_1, \ldots, i_n)}\right] &= n! P((i_1', \ldots, i_n') \text{ is } a \text{ Hamiltonian permuatation}) \\ &= n!/2^{n-1} \end{aligned}$$

Exercise 4. A fair die is continually rolled until an even number has appeared on 10 distinct rolls. Let X_i denote the number of rolls that land on side i. Determine

(a) $E[X_1]$.

(b) $E[X_2]$.

(c) the probability mass function of X_1.

(d) the probability mass function of X_2.

Solution 4.

(a):

Denote I_j be the number of 1 between $(j-1)$th even and jth even.

If we let the expectation conditioning on whether the first appearance of roll is 1 or an even, then we can have

$$\begin{aligned} E[I_j] &= E[I_j | \text{one before even}] P(\text{one before even}) + E[I_j | \text{even before one}] P(\text{even before one}) \\ &= E[I_j | \text{one before even}] \cdot \frac{1}{4} + E[I_j | \text{even before one}] \cdot \frac{3}{4} \\ &= (E[I_j] + 1) \cdot \frac{1}{4} + 0 \end{aligned}$$

where $P(\text{one before even})$ is given by the following equation:

$$\begin{aligned} P(\text{one before even}) &= P(\text{one before } 3 \text{ and } 5) + P(\text{one after } 3 \text{ and } 5) \\ &= \frac{1}{6} + \frac{2}{6} P(\text{one before even}) \end{aligned}$$

Thus $E[I_j] = \frac{1}{3}$, then take the sum of j

$$\begin{aligned} E[X_1] &= E\left[\sum_{j=1}^{10} I_j\right] \\ &= \frac{10}{3} \end{aligned}$$

(b):

Denote Y_j be the number of 1 between $(j-1)$th even and jth even.

If we let the expectation conditioning on whether the appearance of even number is 2, then we can have

$$\begin{aligned} E[Y_j] &= E[Y_j|\text{even num is 2}]P(\text{even num is 2}) + E[Y_j|\text{even num is not 2}]P(\text{even num is not 2}) \\ &= E[Y_j|\text{even num is 2}]\cdot\frac{1}{3} \\ &= 1\cdot\frac{1}{3} \end{aligned}$$

Since even number can only appear once, then $E[Y_j] = \frac{1}{3}$. So take the sum of j

$$\begin{aligned} E[X_2] &= E\left[\sum_{j=1}^{10} Y_j\right] \\ &= \frac{10}{3} \end{aligned}$$

and we have the final result.

(c):

Since $X_1 = \sum_{j=1}^{10} I_j$, then $P(X_1 = i) = P(\sum_{j=1}^{10} I_j = i)$.

But note that $P(\text{even before one}) + P(\text{one before even}) = 1$ and $I_j \in \{0, 1, 2, ...\}$, then $I_j + 1 \sim \text{Ge}\big(\frac{3}{4}\big)$.

Therefore $\sum_{j=1}^{10} I_j + 10 \sim \text{Nb}\big(\frac{3}{4}, 10\big)$, then

$$P(X_1 = i) = P(X_1 + 10 = i + 10) = C_{i+9}^{i}\left(\frac{3}{4}\right)^{10}\left(\frac{1}{4}\right)^{i}$$

(d):

The similar reason as (c), $X_2 \sim b\big(10, \frac{1}{3}\big) \Longrightarrow P(X_2 = i) = C_{10}^{i}\big(\frac{1}{3}\big)^{i}\big(\frac{2}{3}\big)^{10-i}$.

Exercise 5. Let $X_1, ... X_n$ be independent and identically distributed continuous random variables having distribution F. Let X_{in} denote the ith smallest of $X_1, ..., X_n$ and let F_{in} be its distribution function. Show that

(a) $F_{in}(x) = F(x)\, F_{i-1n-1}(x) + \bar{F}(x)\, F_{in-1}(x)$

(b) $F_{in-1}(x) = \frac{i}{n} F_{i+1n}(x) + \frac{n-i}{n} F_{in}(x)$

Solution 5.

(a):

$F_{in}(x) = P(X_{in} \le x) = P(X_{in} \le x | X_n \le x)P(X_n \le x) + P(X_{in} \le x | X_n > x)P(X_n > x)$

Since $X_{in} \leq x \Longleftrightarrow$ at least we have i random variables $\leq x$, then conditioning on $X_n \leq x$ means that we can have at least $i-1$ random variables$\leq x$.

Therefore, $P(X_{in} \leq x | X_n \leq x) = P(X_{i-1n-1} \leq x)$ and $P(X_{in} \leq x | X_n > x) = P(X_{in-1} \leq x)$.

Then we can have $F_{in}(x) = F(x)\, F_{i-1n-1}(x) + \bar{F}(x)\, F_{in-1}(x)$, where $\bar{F}(x) = 1 - F(x)$.

(b):

$F_{in-1}(x) = P(X_{in-1} \leq x) = P(X_{in-1} \leq x | X_n \leq X_{in}) P(X_n \leq X_{in}) + P(X_{in-1} \leq x | X_n > X_{in}) P(X_n > X_{in})$

where $P(X_n \leq X_{in})$ is the probability of event that X_n is among the i smallest of $X_1, \ldots, X_n$.

Thus $P(X_n \leq X_{in}) = \frac{i}{n}$.

And if $X_n \leq X_{in}$, then at least $i+1$ random variables $\leq x$ in $X_1, \ldots, X_n$.

Therefore $P(X_{in-1} \leq x | X_n \leq X_{in}) = P(X_{i+1n} \leq x)$, and we can have the final result.

Exercise 6. A coin, which lands on heads with probability p, is continually flipped. Compute the expected number of flips that are made until a string of r heads in a row is obtained.

Solution 6.

Define $X_i = \begin{cases} 1, \text{the coin lands on head} \\ 0, \text{else} \end{cases}$, then $P(X_i = 1) = p$, $P(X_i = 0) = 1 - p$ and X_i are mutually independent.

Define N be the number of flips that are made until a string of r heads in a row is obtained, and define T be the number of head untill a tail lands. Therefore,

$$E[N|T = k] = \begin{cases} E[N] + k, k \leq r \\ r, k > r \end{cases}$$

Hence by LIE,

$$\begin{aligned} E[N] &= E[E[N|T]] \\ &= \sum_{k=1}^{r} (E[N] + k) P(T = k) + \sum_{k=r+1}^{\infty} r P(T = k) \\ &= \sum_{k=1}^{r} (E[N] + k)(1 - p) p^{k-1} + \sum_{k=r+1}^{\infty} r(1 - p) p^{k-1} \\ &= (1 - p) \frac{1 - p^r}{1 - p} E[N] + (1 - p) \sum_{k=1}^{r} k p^{k-1} + r(1 - p) \frac{p^r}{1 - p} \\ &= (1 - p^r) E[N] + \frac{1 - p^r - r p^r + r p^{r+1}}{1 - p} + \frac{r p^r - r p^{r+1}}{1 - p} \\ &= (1 - p^r) E[N] + \frac{1 - p^r}{1 - p} \end{aligned}$$

Therefore, $E[N] = \frac{1 - p^r}{(1 - p) p^r}$.

Exercise 7. (A Continuous Random Packing Problem) Consider the interval $(0,x)$ and suppose that we pack in this interval random unit intervals—whose left-hand points are all uniformly distributed over $(0,x-1)$-as follows. Let the first such random interval be I_1. If $I_1,\dots,I_k$ have already been packed in the interval, then the next random unit interval will be packed if it does not intersect any of the intervals $I_1,\dots,I_k$, and the interval will be denoted by I_{k+1}. If it does intersect any of the intervals $I_1,\dots,$ I_k, we disregard it and look at the next random interval The procedure is continued until there is no more room for additional unit intervals (that is, all the gaps between packed intervals are smaller than 1). Let $N(x)$ denote the number of unit intervals packed in $[0,x]$ by this method.

Let $M(x)=E[N(x)]$. Show that M satisfies

$$M(x)=0, \quad x<1,$$
$$M(x)=\frac{2}{x-1}\int_0^{x-1} M(y)\,dy+1, \quad x>1$$

Solution 7.

$M(x)=0, x<1$ is very easy to be seen, for a length $r\in(k,k+1)$ interval at most has k such unit intervals.

Let us prove the equality when $x>1$. Denote interval $I_1's$ left-hand point is y, then

$$\begin{aligned}
E[N(x)] &= [E[N(x)|y]] \\
&= \int_0^{x-1}[1+N(y)+N(x-y-1)]\frac{1}{x-1}dy \\
&\overset{x-y-1=t}{=} 1+\frac{1}{x-1}\int_0^{x-1}N(y)dy+\frac{1}{x-1}\int_{x-1}^{0}N(t)d(-t) \\
&= 1+\frac{2}{x-1}\int_0^{x-1}N(y)dy
\end{aligned}$$

Exercise 8. Verify the formulas given for the mean and variance of an exponential random variable.

Solution 8.

Suppose $X\sim \text{Exp}(\lambda)$, then its' pdf is $p(x)=\frac{1}{\lambda}e^{-\frac{1}{\lambda}x}I_{(x>0)}$.

Then the moment generating function is $M(t)=E[e^{tX}]=\int\frac{1}{\lambda}e^{-\frac{1}{\lambda}x}I_{(x>0)}dx=\frac{1}{1-\lambda t}$.

Hence, we can have

$$\begin{aligned}
M'(t) &= \frac{\lambda}{(1-\lambda t)^2} \\
M''(t) &= \frac{2\lambda^2}{(1-\lambda t)^3}
\end{aligned}$$

Thus $E[X]=M'(0)=\lambda$, $E[X^2]=M''(0)=2\lambda^2 \Longrightarrow \text{Var}(X)=2\lambda^2-\lambda^2=\lambda^2$.

Exercise 9. If $X_1,X_2,\dots,X_n$ are independent and identically distributed exponential random variables with parameter λ, show that $\sum_1^n X_i$ has a gamma distribution with parameters (n,λ) That is, show that the density function of $\sum_1^n X_i$ is given by

$$f(t)=\lambda^{-1}e^{-\frac{t}{\lambda}}(\frac{t}{\lambda})^{n-1}/(n-1)!, \quad t\geq 0$$

Solution 9.

By the property of moment generating function, $M_f(t) = \left(\frac{1}{1-\lambda t}\right)^n$. Thus the Laplace transform is $\left(\frac{1}{1+\lambda t}\right)^n$.

So if we want to caculate what is f, we need a Laplace inverse transform:

$$\begin{aligned}
\mathcal{L}^{-1}\left[\left(\frac{1}{1+\lambda t}\right)^n\right] &= \mathrm{Res}\left(\frac{e^{st}}{(1+\lambda s)^n}; -\frac{1}{\lambda}\right) \\
&= \mathrm{Res}\left(\frac{1}{\lambda^n}\frac{e^{st}}{\left(\frac{1}{\lambda}+s\right)^n}; -\frac{1}{\lambda}\right) \\
&= \frac{1}{(n-1)!}\frac{1}{\lambda^n}\frac{d^{n-1}}{ds^{n-1}}e^{st}|_{s=-\frac{1}{\lambda}} \\
&= t^{n-1}\frac{1}{\lambda^n}e^{-\frac{s}{\lambda}}\frac{1}{(n-1)!}
\end{aligned}$$

which is just the pdf of Gamma distribution.

Exercise 10. If X and Y are independent exponential random variables with respective means λ_1 and λ_2, compute the distribution of $Z = \min(X, Y)$. What is the conditional distribution of Z given that $Z = X$?

Solution 10.

$F_Z(z) = P(Z \le z) = P(\min(X,Y) \le z) = 1 - P(X > z, Y > z) = 1 - P(X > z)P(Y > z)$

and $P(X > z) = 1 - F_X(z)$, $P(Y > z) = 1 - F_Y(z)$.

Then $F_Z(z) = 1 - (1 - F_X(z))(1 - F_Y(z)) = 1 - \left(1 - e^{-\frac{1}{\lambda_1}z}\right)\left(1 - e^{-\frac{1}{\lambda_2}z}\right)$

$P(Z \le z | Z = X) = 1 - P(Z > z | Z = X)$, where $P(Z > z | Z = X) = \dfrac{P(X > z | X \le Y)}{P(X \le Y)}$.

And $P(X > z | X \le Y) = \int_z^\infty P(X \le Y | X = x) p_X(x) dx = \int_z^\infty e^{-\frac{1}{\lambda_2}x}\frac{1}{\lambda_1}e^{-\frac{1}{\lambda_1}x}dx = \frac{\lambda_2}{\lambda_1+\lambda_2}e^{-\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)z}$

Then we can have $P(Z > z | Z = X) = \dfrac{P(X > z | X \le Y)}{P(X \le Y)} = \dfrac{\frac{\lambda_2}{\lambda_1+\lambda_2}e^{-\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)z}}{\frac{\lambda_2}{\lambda_1+\lambda_2}} = e^{-\left(\frac{1}{\lambda_1}+\frac{1}{\lambda_2}\right)z}$,

where $P(X \le Y) = \int_0^\infty \int_0^y \frac{1}{\lambda_1\lambda_2}e^{-\left(\frac{1}{\lambda_1}x+\frac{1}{\lambda_2}y\right)}dx\,dy = \frac{\lambda_2}{\lambda_1+\lambda_2}$.

Exercise 11. If X_1 and X_2 are independent nonnegative continuous random variables, and their pdf are strictly increasing, show that

$$P\left\{X_1 < X_2 | \min(X_1, X_2) = t\right\} = \frac{\lambda_1(t)}{\lambda_1(t) + \lambda_2(t)},$$

where $\lambda_i(t)$ is the failure rate function of X_i.

Solution 11.

Since their pdf are sttictly increasing, then $P(X_1 = X_2 = t) = 0$, for some constant t.

Then

$$
\begin{aligned}
P\{X_1 < X_2 | \min(X_1, X_2) = t\} &= \frac{P\{X_1 < X_2, \min(X_1, X_2) = t\}}{P\{\min(X_1, X_2) = t\}} \\
&= \frac{P\{X_1 = t, X_2 > t\}}{P\{X_1 = t, X_2 > t\} + P\{X_2 = t, X_1 > t\}} \\
&= \frac{P\{X_1 = t\} P\{X_2 > t\}}{P\{X_1 = t\} P\{X_2 > t\} + P\{X_2 = t\} P\{X_1 > t\}}
\end{aligned}
$$

Here by the definition of failure function, we can have $\lambda_1(t) = \frac{P(X_1 = t)}{1 - F_1(t)}$ and $\lambda_2(t) = \frac{P(X_2 = t)}{1 - F_2(t)}$. Then $P\{X_1 < X_2 | \min(X_1, X_2) = t\} = \dfrac{\lambda_1(t)}{\lambda_1(t) + \lambda_2(t)}$.

Exercise 12. Use the Markov inequality to show that $e^{-n} \leq \frac{n!}{n^n}$, for all $n \geq 1$.

Solution 12.

Suppose $X \sim \text{Exp}(1)$, then by Markov inequality we can have

$$
\begin{aligned}
P(X > n) &= e^{-n} \\
&= P(X^n > n^n) \\
&\leq \frac{E[X^n]}{n^n} \\
&= \frac{\int_0^\infty x^{n+1-1} e^{-x} dx}{n^n} \\
&= \frac{\Gamma(n+1)}{n^n} \\
&= \frac{n!}{n^n}
\end{aligned}
$$

Thus we finish the proof.

Exercise 13. Consider a particle that moves along the set of integers in the following manner. If it is presently at i then it next moves to $i+1$ with probability p and to $i-1$ with probability $1-p$. Starting at 0 , let α denote the probability that it ever reaches 1.

(a) Argue that

$$\alpha = p + (1-p)\alpha^2.$$

(b) Show that

$$\alpha = \begin{cases} 1 & \text{if } p \geq 1/2 \\ p/(1-p) & \text{if } p < 1/2 \end{cases}$$

(c) Find the probability that the particle ever reaches $n, n > 0$.

(d) Suppose that $p < 1/2$ and also that the particle eventually reaches $n, n > 0$. If the particle is presently at $i, i < n$, and n has not yet been reached, show that the particle will next move to $i+1$ with probability $1-p$ and to $i-1$ with probability p. That is, show that

$$P\{\text{next at } i+1 | \text{at } i \text{ and will reach } n\} = 1 - p$$

(Note that the roles of p and $1-p$ are interchanged when it is given that n is eventually reached)

Solution 13.

Let $i \rightarrow j$ be the event that the particle moves from i to j in one step. Let $i \Rightarrow j$ be the event that the particle ever reaches j starting i. Conditioning on the random variable denoting the first movements of the particle, then (a):

$$\begin{aligned}\alpha =& P(0 \Rightarrow 1)\\ =& P(0 \rightarrow 1)\, P(1 \Rightarrow 1) + P(0 \rightarrow -1) P(-1 \Rightarrow 1)\\ =& p \times 1 + (1-p) P(-1 \Rightarrow 1)\\ =& p + (1-p) P(-1 \Rightarrow 0, 0 \Rightarrow 1)\\ =& p + (1-p) P(-1 \Rightarrow 0) P(0 \Rightarrow 1)\\ =& p + (1-p) [P(0 \Rightarrow 1)]^2\\ =& p + (1-p)\, \alpha^2\end{aligned}$$

(b):

Solve the quadratic equation and have the result.

Note that $\frac{p}{1-p} < 1$ implies $p < 1/2$, and SLLN tells us $S_n \longrightarrow -\infty$ a.s, if $p < 1/2$, where $S_n = \sum_{i=1}^{n} X_i$, and X_i is the move of particle at i step.

Thus

$$\alpha = \begin{cases} 1 & \text{if } p \geq 1/2 \\ p/(1-p) & \text{if } p < 1/2 \end{cases}$$

(c):

$$\begin{aligned} P(0 \Rightarrow n) &= P(n-1 \Rightarrow n) P(n-2 \Rightarrow n-1) \cdots P(0 \Rightarrow 1)\\ &= [P(0 \Rightarrow 1)]^n\\ &= \alpha^n \end{aligned}$$

(d):

$$\begin{aligned} P(\text{next at } i+1 | \text{at } i \text{ \& will reach } n) &= P(i \rightarrow i+1 | i \Rightarrow n)\\ &= \frac{P(i \rightarrow i+1, i \Rightarrow n)}{P(i \Rightarrow n)}\\ &= \frac{P(i \Rightarrow n | i \rightarrow i+1) P(i \rightarrow i+1)}{P(i \Rightarrow n)}\\ &= \frac{\alpha^{n-i-1} p}{\alpha^{n-i}}\\ &= \frac{p}{\frac{p}{1-p}}\\ &= 1-p \end{aligned}$$

Exercise 14. In Exercise 13, let $E[T]$ denote the expected time until the particle reaches 1

(a) Show that

$$E[T] = \begin{cases} 1/(2p-1) & \text{if } p > 1/2 \\ \infty & \text{if } p \leq 1/2 \end{cases}$$

(b) Show that, for $p > 1/2$,

$$\text{Var}(T) = \frac{4p(1-p)}{(2p-1)^3}$$

(c) Find the expected time until the particle reaches $n, n>0$.

(d) Find the variance of the time at which the particle reaches $n, n>0$

Solution 14.

Let $T_{(i\Rightarrow j)}$ the number of steps to reach j first time starting i. Then we have an apparent arithmetic like $T_{(-1\Rightarrow 1)} = T_{(-1\Rightarrow 0)} + T_{(0\Rightarrow 1)}$ and a distributional identity like $T_{(-1\Rightarrow 0)} \overset{d}{=} T_{(0\Rightarrow 1)}$. We also know that $T_{(-1\Rightarrow 0)}$ and $T_{(0\Rightarrow 1)}$ are independent because of the independence of every transition. That is, $T_{(-1\Rightarrow 0)}$ and $T_{(0\Rightarrow 1)}$ are iid. Using the notation in the book, $T \equiv T_{(0\Rightarrow 1)}$,

$$
\begin{aligned}
E[T_{(-1\Rightarrow 1)}] =& 2E[T] \\
\mathrm{Var}(T_{(-1\Rightarrow 1)}) =& 2\mathrm{Var}(T)
\end{aligned}
$$

Let the random variable X denote the particle's location after the first move.

(a):

Conditioning on X,

$$
\begin{aligned}
E[T] =& E[E[T|\,X]] \\
=& E[T|\,X=1]P(X=1) + E[T|\,X=-1]P(X=-1) \\
=& 1 \times p + (1 + E[T_{(-1\Rightarrow 1)}])\,(1-p) \\
=& 1 + 2\,(1-p)E[T].
\end{aligned}
$$

Hence, $E[T] = \infty$ if $p \leq 1/2$. If we can show that $E[T] < \infty$ when $p > 1/2$, we obtain in this case that

$$
E[T] = \frac{1}{2\,p-1}.
$$

Now let's show that $E[T] < \infty$ if $p > 1/2$: Let $p^{(n)}$ denotes the probability that the particle reaches 1 by n-transitions starting 0. Then n should be odd. That is, only $p^{(2n+1)}$ is nonzero. Now we have an upper bound on this probability:

$$
p^{(2n+1)} \leq \binom{2\,n}{n} p\,[p\,(1-p)]^n \sim p\frac{[4\,p\,(1-p)]^n}{\sqrt{\pi\,n}}
$$

the last approximation is due to Stirling: $n! \sim n^{n+1/2}e^{-n}\sqrt{2\pi}$.

Since $\sum_{n=1}^{\infty} p\frac{[4\,p\,(1-p)]^n}{\sqrt{\pi\,n}} < \infty$, then $\sum_{n=1}^{\infty} p^{(2n+1)} < \infty$.

Thus $p^{(2n+1)} \longrightarrow 0$ as $n \longrightarrow \infty$.

(b):

Similarly as (a)

$$
\begin{aligned}
E[T^2] &= E[E[T^2|X]] \\
&= E[T^2|X=1]P(X=1) + E[T^2|X=-1]P(X=-1) \\
&= p \times 1 + (1-p)E[1 + T_{(-1\Rightarrow 1)}]^2 \\
&= p + (1-p)E[1 + T_{(-1\Rightarrow 0)} + T_{(0\Rightarrow 1)}]^2 \\
&= p + (1-p)\{E[1 + 4T + 2T^2] + 2(E[T])^2\}
\end{aligned}
$$

then we can have $E[T^2] = \dfrac{1}{2p-1} + \dfrac{4-4p}{(2p-1)^2} + \dfrac{2-2p}{(2p-1)^3}$.

Therefore, $\text{Var}(T) = E[T^2] - (E[T])^2 = \dfrac{4p-4p^2}{(2p-1)^3} = \dfrac{4p(1-p)}{(2p-1)^3}$.

(c):

$T_{(0 \Rightarrow n)} = T_{(0 \Rightarrow 1)} + \cdots + T_{(n-1 \Rightarrow n)} = \sum_{i=1}^{n} T_i$ where T_i are i.i.d having distribution of T. Hence

$$E[T_{(0 \Rightarrow n)}] = n\, E[T].$$

(d):

By the same reasoning as in (c), $\text{Var}(T_{(0 \Rightarrow n)}) = n\, \text{Var}(T)$